On the Three-Terminal Interactive Lossy Source Coding Problem

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***Abstract*—The three-node multiterminal lossy source coding problem is investigated. We derive an inner bound to the general rate-distortion region of this problem which appears to be the natural extension of the seminal work by Kaspi [1] on the interactive two-terminal source coding problem. It is shown that this –rather involved– inner bound contains several rate-distortion regions of some relevant source coding settings. In this way, besides the non-trivial extension of the interactive two terminal problem, our results can be seen as a generalization and hence unification of several previous works in the field.**

I. Introduction

Efficient distributed data compression enormous relevance as larger. Data compression may be the only way to guarantee acceptable levels of performance, e.g., anomaly detection, when energy and link bandwidth are severely limited as in many real world sensor networks. The distributed data collected by different nodes in a network can be highly correlated and this correlation can be exploited at the application layer, e.g., for target localization and tracking or anomaly detection. In such cases cooperative joint data-compression can achieve a better overall rate-distortion tradeoff that can independent compression at each node. Interaction among nodes may take place via distributed/successive refinement source coding, where nodes exchange –interactively– data among themselves over a given number of communication rounds.

The value of interaction for source coding problems was first recognized by Kaspi in his seminal work [1], where the interactive two-terminal lossy source coding problem was introduced and solved under the assumption of a finite number of communication rounds. Although several extensions to this problem exists (e.g. see [2]–[4] and references therein), to the best of our knowledge, a proper generalization of this setting to interactive multiterminal (> 2) lossy source coding has not be studied yet.

In this paper, we consider the three-terminal interactive lossy source coding problem presented in Fig. 1. We have a network composed of 3 nodes which can interact through rate-limited –error free– links. Each node measures the realization of a discrete memoryless source (DMS) and is required to

The work of L. Rey Vega was partially supported by project UBACyT 2002010200250. The work of A. Hero was partially supported by a DIGITEO Chair from 2008 to 2013.

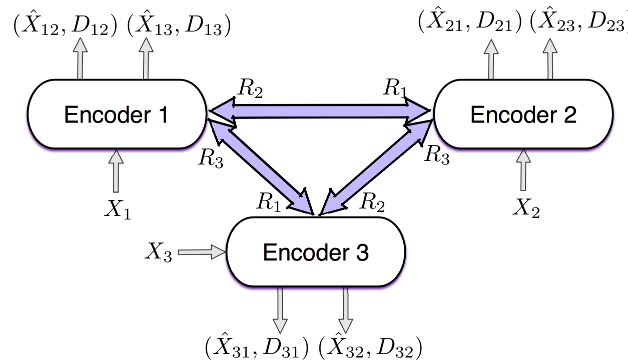

Figure 1: Three-terminal interactive source coding network.

reconstruct the sources from the other terminals with a fidelity criterion. Nodes are allowed to interact by interchanging information over a finite number of communication rounds. After the information exchange phase is over, the nodes try to reconstruct the realization of the sources at the other nodes using all decoded descriptions. We first derive a general achievable region by assuming a finite number of rounds. Then, we show that this inner bound to the rate-distortion region allows us to recover several previous inner bounds and rate-distortion regions of some well-known interactive –as well non-interactive– lossy source coding problems. We summarize the notation. Boldface letters x^n and upper-case letters X^n are used to denote vectors and random vectors of n components, respectively. All alphabets are assumed to be finite. Entropy is denoted by $H(\cdot)$ and mutual information by $I(\cdot;\cdot)$. Let X, Y and V be three random variables on some alphabets with probability distribution p. If $p(x|yv) = p(x|y)$ for each x, y, v, then they form a Markov chain, which is denoted by $X -\!\!\circ\!\!- Y -\!\!\circ\!\!- V$. The set of strong typical sequences is denoted by $\mathcal{T}^n_{[X]\epsilon}$. We simply denote these sets as $\mathcal{T}^n_\epsilon$ when clear from the context. The cardinal of set $\mathcal{A}$ is denoted by $\|\mathcal{A}\|$.

II. Problem Statement and Main Result

A. *Problem definition*

Assume three DMS's with alphabets and pmfs given by $(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3, p_{X_1X_2X_3})$. Consider arbitrary distortion measures: $d_i : \mathcal{X}_i \times \hat{\mathcal{X}}_i \to \mathbb{R}_{\geq 0}, \quad i \in \mathcal{M} \triangleq \{1, 2, 3\}$ where $\{\hat{\mathcal{X}}_i\}$ are finite reconstruction alphabets. We consider the problem of characterizing the rate-distortion region of the interactive source coding setting described in Fig. 1. In this setting, through K rounds of information exchange between the nodes each one of them will attempt to recover a lossy description of the sources that the others nodes observe, e.g., node 1 must reconstruct –while satisfying distortion constraints– the realization of the sources X_2^n and X_3^n observed by nodes 2 and 3. Indeed, this setting can be seen as a generalization of the well-known Kaspi's problem.

Definition 1 (K-step interactive source code): A K-step interactive source code $\big(n, \{f_{x_1}^l, f_{x_2}^l, f_{x_3}^l\}_{l=1}^K, g_{12}, g_{13}, g_{21}, g_{23}, g_{31}, g_{32}\big)$ for the network model in Fig. 1 is defined by a sequence $l \in [1:K]$ of encoder mappings:

$$f_{x_1}^l \colon \mathcal{X}_1^n \times \left(\mathcal{J}_{x_2}^1 \times \mathcal{J}_{x_3}^1 \times \cdots \times \mathcal{J}_{x_2}^{l-1} \times \mathcal{J}_{x_3}^{l-1}\right) \longrightarrow \mathcal{J}_{x_1}^l ,$$
$$f_{x_2}^l \colon \mathcal{X}_2^n \times \left(\mathcal{J}_{x_1}^1 \times \mathcal{J}_{x_3}^1 \times \cdots \times \mathcal{J}_{x_1}^{l} \times \mathcal{J}_{x_3}^{l-1}\right) \longrightarrow \mathcal{J}_{x_2}^l ,$$
$$f_{x_3}^l \colon \mathcal{X}_3^n \times \left(\mathcal{J}_{x_1}^1 \times \mathcal{J}_{x_2}^1 \times \cdots \times \mathcal{J}_{x_2}^{l} \times \mathcal{J}_{x_3}^{l}\right) \longrightarrow \mathcal{J}_{x_3}^l ,$$

with message sets: $\mathcal{J}_{x_i}^l \triangleq \left\{1, 2, \ldots, \mathcal{I}_{x_i}^l\right\}, \quad \mathcal{I}_{x_i}^l \in \mathbb{Z}_{\geq 0}$, and reconstruction mappings:

$$g_{ij} : \mathcal{X}_i^n \times \bigotimes_{m \in \mathcal{M},\ m \neq i} \left(\mathcal{J}_{x_m}^1 \times \cdots \times \mathcal{J}_{x_m}^K\right) \longrightarrow \hat{\mathcal{X}}_j^n,\ i \neq j.$$

The distortion is defined as the average of the per-letter distortion $d_i\left(x^n, y^n\right) \equiv \frac{1}{n}\sum_{m=1}^n d_i(x_m, y_m)$.

Definition 2 (Achievability and rate-distortion region): The tuple of rates and distortions (R_1, R_2, R_3) and $\mathbf{D} \triangleq (D_{12}, D_{13}, D_{21}, D_{23}, D_{31}, D_{32})$, respectively, are $(K, \mathbf{D})$-achievable if $\forall \epsilon > 0$ exists a K-step interactive source code with rates satisfying:

$$\frac{1}{n}\sum_{l=1}^K \log \|\mathcal{J}_{x_i}^l\| \leq R_i + \epsilon,\ i \in \mathcal{M}$$

and whose corresponding mean distortions verify:

$$\mathbb{E}\left[d_i(X_i^n, \hat{X}_{ji}^n)\right] \leq D_{ji} + \epsilon,\ i, j \in \mathcal{M},\ i \neq j\ ,$$

where $\hat{X}_{ij}^n \equiv g_{ij}\left(X_i^n, h_i\left(X_1^n, X_2^n, X_3^n\right)\right) \quad i \neq j \in \mathcal{M}$ with

$$h_i\left(X_1^n, X_2^n, X_3^n\right) \equiv \bigotimes_{m \in \mathcal{M},\ m \neq i} \left(\mathcal{J}_{x_m}^1 \times \cdots \times \mathcal{J}_{x_m}^K\right)\ .$$

The rate-distortion region $\mathcal{R}^K(\mathbf{D})$ is the set of all $\big(R_1, R_2, R_3, K, \mathbf{D}\big)$-achievable tuples. Similarly, the $\mathbf{D}$-achievable region $\mathcal{R}(\mathbf{D})$ is given by $\mathcal{R}(\mathbf{D}) = \mathrm{co}\left(\bigcup_{K=1}^\infty \mathcal{R}^K(\mathbf{D})\right)$, where $\mathrm{co}\,(\cdot)$ denotes the closure of the convex hull.

Remark 1: Using a time-sharing argument it is easy to show that $\mathcal{R}^K(\mathbf{D})$ is closed and convex $\forall K \in \mathbb{Z}_{\geq 0}$.

Remark 2: $\mathcal{R}^K(\mathbf{D})$ depends on the node ordering in the encoding procedure. Above we defined the encoding functions $\left\{f_{x_1}^l, f_{x_2}^l, f_{x_3}^l\right\}_{l=1}^K$ assuming that in each round node 1 acts first, followed by node 2, and finally by node 3, and the process begins again from node 1. For simplicity and without loss of generality, we will assume the *canonical ordering* ($1 \to 2 \to 3$). However, there are $3! = 6$ different orderings that may lead to different rate-distortion regions.

B. Inner bound to the rate-distortion region

We present an achievable rate-region where each node at a given round l will generate descriptions destined to the other nodes based on the realization of its own source, the past descriptions generated by him and the descriptions generated at the other nodes and recovered by him up to the present round. In order to precisely describe the complex rate-region, we need to introduce some definitions. For a set $\mathcal{M}$, let $\mathcal{C}\left(\mathcal{M}\right) = 2^{\mathcal{M}} \setminus \{\mathcal{M}, \emptyset\}$ be the set of all subsets of $\mathcal{M}$ minus $\mathcal{M}$ and the empty set. Denote the auxiliary random variables:

$$U_{i \to S, l}, \quad S \in \mathcal{C}\left(\mathcal{M}\right),\ i \notin S,\ l = 1, \ldots, K. \tag{1}$$

Auxiliary random variables $\{U_{i \to S, l}\}$ will be used to denote the descriptions in node i and at round l and intended to a set of nodes $S \in \mathcal{C}\left(\mathcal{M}\right)$ with $i \notin S$[1]. Similarly, $\{U_{1 \to 2, l}\}$ will be used to denote the descriptions in node 1 at round l and destined only to node 2. Let us also define variables:

$\mathcal{W}_{[i,l]} \equiv$ Common information shared by the three nodes available at node i at round l before encoding

$\mathcal{V}_{[S,l,i]} \equiv$ Private information shared by nodes in $S \in \mathcal{C}\left(\mathcal{M}\right)$ available at node $i \in S$, at round l, before encoding

As an example of the above definitions, we may have:

$$\mathcal{W}_{[1,l]} = \{U_{1 \to 23,k}, U_{2 \to 13,k}, U_{3 \to 12,k}\}_{k=1}^{l-1}$$
$$\mathcal{W}_{[2,l]} = \mathcal{W}_{[1,l]} \cup U_{1 \to 23,l},\ \mathcal{W}_{[3,l]} = \mathcal{W}_{[2,l]} \cup U_{2 \to 13,l}$$
$$\mathcal{V}_{[12,l,1]} = \{U_{1 \to 2,k}, U_{2 \to 1,k}\}_{k=1}^{l-1},\ \mathcal{V}_{[12,l,2]} = \mathcal{V}_{[12,l,1]} \cup U_{1 \to 2,l}$$
$$\mathcal{V}_{[13,l,1]} = \{U_{1 \to 3,k}, U_{3 \to 1,k}\}_{k=1}^{l-1},\ \mathcal{V}_{[13,l,3]} = \mathcal{V}_{[13,l,1]} \cup U_{1 \to 3,l}$$
$$\mathcal{V}_{[23,l,2]} = \{U_{2 \to 3,k}, U_{3 \to 2,k}\}_{k=1}^{l-1},\ \mathcal{V}_{[23,l,3]} = \mathcal{V}_{[23,l,2]} \cup U_{2 \to 3,l}$$

Theorem 1 (Inner bound): Let $\bar{\mathcal{R}}^K(\mathbf{D})$ be the set of tuples:

$$R_1 \geq \sum_{l=1}^K \left(R_{1\to 23}^{(l)} + R_{1\to 2}^{(l)} + R_{1\to 3}^{(l)}\right) \tag{2}$$

$$R_2 \geq \sum_{l=1}^K \left(R_{2\to 13}^{(l)} + R_{2\to 1}^{(l)} + R_{2\to 3}^{(l)}\right) \tag{3}$$

$$R_3 \geq \sum_{l=1}^K \left(R_{3\to 12}^{(l)} + R_{3\to 1}^{(l)} + R_{3\to 2}^{(l)}\right) \tag{4}$$

$$R_1 + R_2 \geq \sum_{l=1}^K \Big(R_{1\to 23}^{(l)} + R_{2\to 13}^{(l)} + R_{1\to 3}^{(l)} + R_{2\to 3}^{(l)} + R_{1\to 2}^{(l)} + R_{2\to 1}^{(l)}\Big) \tag{5}$$

$$R_1 + R_3 \geq \sum_{l=2}^K \Big(R_{1\to 23}^{(l)} + R_{3\to 12}^{(l-1)} + R_{1\to 2}^{(l)} + R_{3\to 2}^{(l-1)} \tag{6}$$

$$+ R_{1\to 3}^{(l)} + R_{3\to 1}^{(l)}\Big) + \Big(R_{1\to 23}^{(1)} + R_{1\to 2}^{(1)} + R_{1\to 3}^{(1)} + R_{3\to 1}^{(1)} + R_{3\to 12}^{(K)} + R_{3\to 2}^{(K)}\Big) \tag{7}$$

$$R_2 + R_3 \geq \sum_{l=1}^K \Big(R_{2\to 13}^{(l)} + R_{3\to 12}^{(l)} + R_{2\to 1}^{(l)} + R_{3\to 1}^{(l)} + R_{2\to 3}^{(l)} + R_{3\to 2}^{(l)}\Big) \tag{8}$$

[1] $U_{1\to 23,l}$ is to be used to generate the descriptions in node 1 and round l and destined to be transmitted to nodes 2 and 3.

Then, the rate-distortion region satisfies:

$$\text{co}\left(\bigcup_{p\in\mathcal{P}(\mathbf{D})} \bar{\mathcal{R}}^K(\mathbf{D})\right) \subseteq \mathcal{R}^K(\mathbf{D}) \ , \tag{9}$$

where

$$R_{1\to23}^{(l)} > \max\{I(X_1;U_{1\to23,l}|X_3\mathcal{W}_{[1,l]}) - I(X_3;U_{2\to13,l}|\mathcal{W}_{[2,l]}), I(X_1;U_{1\to23,l}|X_2\mathcal{W}_{[1,l]})\} \tag{10}$$

$$R_{2\to13}^{(l)} > \max\{I(X_2;U_{2\to13,l}|X_1\mathcal{W}_{[2,l]}) - I(X_1;U_{3\to12,l}|\mathcal{W}_{[3,l]}), I(X_2;U_{2\to13,l}|X_3\mathcal{W}_{[2,l]})\} \tag{11}$$

$$R_{3\to12}^{(l)} > \max\{I(X_3;U_{3\to12,l}|X_2\mathcal{W}_{[3,l]}) - I(X_2;U_{1\to23,l}|\mathcal{W}_{[1,l+1]}), I(X_3;U_{3\to12,l}|X_1\mathcal{W}_{[3,l]})\} \tag{12}$$

$$R_{1\to23}^{(l)} + R_{2\to13}^{(l)} > I(X_1X_2;U_{1\to23,l}U_{2\to13,l}|X_3\mathcal{W}_{[1,l]}) \tag{13}$$

$$R_{2\to13}^{(l)} + R_{3\to12}^{(l)} > I(X_2X_3;U_{2\to13,l}U_{3\to12,l}|X_1\mathcal{W}_{[2,l]}) \tag{14}$$

$$R_{3\to12}^{(l-1)} + R_{1\to23}^{(l)} > I(X_1X_3;U_{1\to23,l}U_{3\to12,l-1}|X_2\mathcal{W}_{[3,l-1]}) \tag{15}$$

$$R_{3\to2}^{(l-1)} > I(X_3;U_{3\to2,l-1}|X_2\mathcal{W}_{[1,l]}\mathcal{V}_{[23,l-1,3]}\mathcal{V}_{[12,l,1]}) - I(U_{3\to2,l-1};U_{1\to23,l}U_{1\to2,l}|X_2\mathcal{W}_{[1,l]}\mathcal{V}_{[23],l-1,3}\mathcal{V}_{[12,l,1]}) \tag{16}$$

$$R_{1\to2}^{(l)} > I(X_1;U_{1\to2,l}|X_2\mathcal{W}_{[2,l]}\mathcal{V}_{[23,l,2]}\mathcal{V}_{[12,l,1]}) \tag{17}$$

$$R_{1\to2}^{(l)} + R_{3\to2}^{(l-1)} > I(X_1;U_{1\to2,l}|X_2\mathcal{W}_{[2,l]}\mathcal{V}_{[23,l,2]}\mathcal{V}_{[12,l,1]}) + I(X_3;U_{3\to2,l-1}|X_2\mathcal{W}_{[1,l]}\mathcal{V}_{[23,l-1,3]}\mathcal{V}_{[12,l,1]}) - I(U_{3\to2,l-1};U_{1\to23,l}|X_2\mathcal{W}_{[1,l]}\mathcal{V}_{[23],l-1,3}\mathcal{V}_{[12,l,1]}) \tag{18}$$

$$R_{1\to3}^{(l)} > I(X_1;U_{1\to3,l}|X_3\mathcal{W}_{[2,l]}\mathcal{V}_{[23,l,2]}\mathcal{V}_{[13,l,1]}) - I(U_{1\to3,l};U_{2\to13,l}U_{2\to3,l}|X_3\mathcal{W}_{[2,l]}\mathcal{V}_{[23,l,2]}\mathcal{V}_{[13,l,1]}) \tag{19}$$

$$R_{2\to3}^{(l)} > I(X_2;U_{2\to3,l}|X_3\mathcal{W}_{[3,l]}\mathcal{V}_{[23,l,2]}\mathcal{V}_{[13,l,3]}) \tag{20}$$

$$R_{1\to3}^{(l)} + R_{2\to3}^{(l)} > I(X_1;U_{1\to3,l}|X_3\mathcal{W}_{[2,l]}\mathcal{V}_{[23,l,2]}\mathcal{V}_{[13,l,1]}) + I(X_2;U_{2\to3,l}|X_3\mathcal{W}_{[3,l]}\mathcal{V}_{[23,l,2]}\mathcal{V}_{[13,l,3]}) - I(U_{1\to3,l};U_{2\to13,l}|X_3\mathcal{W}_{[2,l]}\mathcal{V}_{[23,l,2]}\mathcal{V}_{[13,l,1]}) \tag{21}$$

$$R_{2\to1}^{(l)} > I(X_2;U_{2\to1,l}|X_1\mathcal{W}_{[3,l]}\mathcal{V}_{[12,l,2]}\mathcal{V}_{[13,l,3]}) - I(U_{2\to1,l};U_{3\to12,l}U_{3\to1,l}|X_1\mathcal{W}_{[3,l]}\mathcal{V}_{[12,l,2]}\mathcal{V}_{[13,l,3]}) \tag{22}$$

$$R_{3\to1}^{(l)} > I(X_3;U_{3\to1,l}|X_1\mathcal{W}_{[1,l+1]}\mathcal{V}_{[12,l+1,1]}\mathcal{V}_{[13,l,3]}) \tag{23}$$

$$R_{2\to1}^{(l)} + R_{3\to1}^{(l)} > I(X_2;U_{2\to1,l}|X_1\mathcal{W}_{[3,l]}\mathcal{V}_{[12,l,2]}\mathcal{V}_{[13,l,3]}) + I(X_3;U_{3\to1,l}|X_1\mathcal{W}_{[1,l+1]}\mathcal{V}_{[12,l+1,1]}\mathcal{V}_{[13,l,3]}) - I(U_{2\to1,l};U_{3\to12,l}|X_1\mathcal{W}_{[3,l]}\mathcal{V}_{[12,l,2]}\mathcal{V}_{[13,l,3]}) \tag{24}$$

and $\mathcal{P}(\mathbf{D},K)$ denote the set of all joint probability measures satisfying the following Markov chains for every $l \in [1:K]$:

1) $U_{1\to23,l} -\!\!\circ\!\!- (X_1\mathcal{W}_{[1,l]}) -\!\!\circ\!\!- (X_2X_3\mathcal{V}_{[12,l,1]}\mathcal{V}_{[13,l,1]}\mathcal{V}_{[23,l,2]})$
2) $U_{1\to2,l} -\!\!\circ\!\!- (X_1\mathcal{W}_{[2,l]}\mathcal{V}_{[12,l,1]}) -\!\!\circ\!\!- (X_2X_3\mathcal{V}_{[13,l,1]}\mathcal{V}_{[23,l,2]})$
3) $U_{1\to3,l} -\!\!\circ\!\!- (X_1\mathcal{W}_{[2,l]}\mathcal{V}_{[13,l,1]}) -\!\!\circ\!\!- (X_2X_3\mathcal{V}_{[12,l,2]}\mathcal{V}_{[23,l,2]})$
4) $U_{2\to13,l} -\!\!\circ\!\!- (X_2\mathcal{W}_{[2,l]}) -\!\!\circ\!\!- (X_1X_3\mathcal{V}_{[12,l,2]}\mathcal{V}_{[13,l,3]}\mathcal{V}_{[23,l,2]})$
5) $U_{2\to1,l} -\!\!\circ\!\!- (X_2\mathcal{W}_{[3,l]}\mathcal{V}_{[12,l,2]}) -\!\!\circ\!\!- (X_1X_3\mathcal{V}_{[13,l,3]}\mathcal{V}_{[23,l,2]})$
6) $U_{2\to3,l} -\!\!\circ\!\!- (X_2\mathcal{W}_{[3,l]}\mathcal{V}_{[23,l,2]}) -\!\!\circ\!\!- (X_1X_3\mathcal{V}_{[12,l+1,1]}\mathcal{V}_{[13,l,3]})$
7) $U_{3\to12,l} -\!\!\circ\!\!- (X_3\mathcal{W}_{[3,l]}) -\!\!\circ\!\!- (X_1X_2\mathcal{V}_{[12,l+1,1]}\mathcal{V}_{[13,l,3]}\mathcal{V}_{[23,l,3]})$
8) $U_{3\to1,l} -\!\!\circ\!\!- (X_3\mathcal{W}_{[1,l+1]}\mathcal{V}_{[13,l,3]}) -\!\!\circ\!\!- (X_1X_2\mathcal{V}_{[12,l+1,1]}\mathcal{V}_{[23,l,3]})$
9) $U_{3\to2,l} -\!\!\circ\!\!- (X_3\mathcal{W}_{[1,l+1]}\mathcal{V}_{[23,l,3]}) -\!\!\circ\!\!- (X_1X_2\mathcal{V}_{[12,l+1,1]}\mathcal{V}_{[13,l+1,1]})$

such that there exist mappings:

$$\tilde{g}_{ji}\left(X_1, \mathcal{V}_{[ji,K+1,1]}, W_{[1,K+1]}\right) = \hat{X}_{ji} \tag{25}$$

with $\mathbb{E}\left[d_i(X_i,\hat{X}_{ji})\right] \leq D_{ji}$ for each $i,j \in \mathcal{M}$ and $i \neq j$.

III. Cases of Special Interest and Related Work

Several inner bounds and rate-distortion regions on multiterminal source coding problems can be derived by specializing the inner bound (9). Below we summarize only a few of them for lack of space.

1) Distributed source coding with side information [5], [6]: Consider the distributed source coding problem where two nodes encode separately sources X_1 and X_2 to rates (R_1, R_2) and a decoder by using side information X_3 must reconstruct both sources with average distortion less than D_1 and D_2, respectively. By considering only one round/way information exchange from nodes 1 and 2 (the encoders) to node 3 (the decoder), the results in [5], [6] can be recovered as a special case of the inner bound (9). Specifically, we set: $U_{1\to23,l} = U_{2\to13,l} = U_{3\to12,l} = U_{1\to2,l} = U_{2\to1,l} = U_{3\to1,l} = U_{3\to2,l} = \varnothing$, $\forall l$ and $U_{1\to3,l} = U_{2\to3,l} = \varnothing$, $\forall l > 1$. In this case, the above Markov chains imply: $U_{1\to3,1} -\!\!\circ\!\!- X_1 -\!\!\circ\!\!- (X_2X_3U_{2\to3,1})$ and $U_{2\to3,1} -\!\!\circ\!\!- X_2 -\!\!\circ\!\!- (X_1X_3U_{1\to3,1})$ and thus the rate-distortion region (9) reduces to the results in [6]

$$R_1 > I(X_1;U_{1\to3,1}|X_3U_{2\to3,1}), R_2 > I(X_2;U_{2\to3,1}|X_3U_{1\to3,1})$$
$$R_1 + R_2 > I(X_1X_2;U_{1\to3,1}U_{2\to3,1}|X_3) \ .$$

2) Source coding with side information at 2-decoders [7], [8]: Consider the setting where one encoder X_1 transmits descriptions to two decoders with different side informations (X_2, X_3) and distortion requirements D_2 and D_3. Again we consider only one way/round information exchange from node 1 (the encoder) to nodes 2 and 3 (the decoders). In this case, we set: $U_{2\to13,l} = U_{3\to12,l} = U_{2\to1,l} = U_{3\to1,l} = U_{3\to2,l} = U_{2\to3,l} = \varnothing$, $\forall l$ and $U_{1\to23,l} = U_{1\to23,l} = U_{1\to2,l} = U_{1\to3,l} = \varnothing$, $\forall l > 1$. The above Markov chains imply $(U_{1\to23,1}U_{1\to2,1}U_{1\to3,1}) -\!\!\circ\!\!- X_1 -\!\!\circ\!\!- (X_2X_3)$ and thus the rate-distortion region (9) reduces to the results in [7], [8]

$$R_1 > \max\{I(X_1;U_{1\to23,1}|X_2), I(X_1;U_{1\to23,1}|X_3)\} + I(X_1;U_{1\to2,1}|X_2U_{1\to23,1}) + I(X_1;U_{1\to3,1}|X_3U_{1\to23,1}).$$

3) Two terminal interactive source coding [1]: Our inner bound (9) is basically the generalization of the two terminal problem to the three-terminal setting. Assume only two encoders-decoders X_1 and X_2 which must reconstruct the other terminal source 3 with distortion constraints D_1 and D_2, and after K rounds of information exchange. Let us set: $U_{1\to23,l} = U_{2\to13,l} = U_{3\to12,l} = U_{1\to3,l} = U_{3\to1,l} = U_{2\to3,l} = U_{3\to2,l} = \varnothing$, $\forall l$ and $X_3 = \varnothing$. The Markov chains become $U_{1\to2,l} -\!\!\circ\!\!- (X_1\mathcal{V}_{[12,l,1]}) -\!\!\circ\!\!- X_2$ and $U_{2\to1,l} -\!\!\circ\!\!-$

$(X_2\mathcal{V}_{[12,l,2]}) -\!\!\circ\!\!- X_2$ for $l\in[1:K]$ and thus the rate-distortion region (9) to the results in [1]

$$R_1 > I(X_1;\mathcal{V}_{[12,K+1,1]}|X_2),\ R_2 > I(X_2;\mathcal{V}_{[12,K+1,1]}|X_1).$$

4) Two terminal interactive source coding with a helper [2]: Consider now two encoders/decoders, namely X_2 and X_3, that must reconstruct the other terminal source with distortion constraints D_2 and D_3, respectively, using K communication rounds. Assume also that another encoder X_1 provides both nodes $(2,3)$ with a common description before beginning the information exchange and then remains silent. Such common description can be exploited as coded side information. Let us set: $U_{2\to 13,l} = U_{3\to 12,l} = U_{1\to 3,l} = U_{1\to 2,l} = U_{1\to 3,l} = U_{2\to 1,l} = U_{3\to 1,l} = \varnothing$, $\forall l$ and $U_{1\to 23,l}=\varnothing$, $\forall l>1$. The Markov chains reduce to:

$$\begin{aligned}
&U_{1\to 23,1} -\!\!\circ\!\!- X_1 -\!\!\circ\!\!- (X_2X_3)\ ,\\
&U_{2\to 3,l} -\!\!\circ\!\!- (X_2U_{1\to 23,1}\mathcal{V}_{[23,l,2]}) -\!\!\circ\!\!- (X_1X_3)\ ,\\
&U_{3\to 2,l} -\!\!\circ\!\!- (X_3U_{1\to 23,1}\mathcal{V}_{[23,l,3]}) -\!\!\circ\!\!- (X_1X_2)\ .
\end{aligned}$$

The rate-distortion region (9) reduces to

$$\begin{aligned}
&R_1 > \max\{I(X_1;U_{1\to 23,1}|X_2), I(X_1;U_{1\to 23,1}|X_3)\}\\
&R_2 > I(X_2;\mathcal{V}_{[23,K+1,2]}|X_3U_{1\to 23,1})\\
&R_3 > I(X_3;\mathcal{V}_{[23,K+1,2]}|X_2U_{1\to 23,1}),
\end{aligned}$$

This region contains the region in [2]. As in that paper it is further assumed (in order to have a converse result) that $X_1 -\!\!\circ\!\!- X_3 -\!\!\circ\!\!- X_2$, the value of R_1 satisfies $R_1 > I(X_1;U_{1\to 23,1}|X_2)$. Obviously, with the same extra Markov chain we obtain the same limiting value for R_1.

IV. Proof of the Inner Bound in Theorem 1

A. Main idea behind the proof

We first provide the basic idea of the *random coding* scheme that achieves the rate-region in Theorem 1 for the case of K communication rounds.

Assume that all codebooks are randomly generated and revealed to all the nodes before the information exchange begins and consider the encoding ordering given by $1\to 2\to 3$ so that we begin at round $l=1$ in node 1. From the observation of the source X_1^n, node 1 generates a set of descriptions for each of the other nodes in the networks. Then, node 2 tries to recover the descriptions destined to it (using X_2^n as side information) and generates its own descriptions, based on source X_2^n and the recovered descriptions from node 1. The same process goes on until node 3, which tries to recover the descriptions generated by node 1 and node 2 destined to it –using X_3^n as side information–, and then generates its own descriptions destined to nodes 1 and 2. Finally, node 1 tries to recover all the descriptions destined to it generated by nodes 2 and 3. After this, round $l=1$ is over, and round $l=2$ begins with node 1 generating new descriptions using X_1^n, its encoding history (from previous round) and the recovered descriptions from the other nodes. The process continues in a similar manner until we reach round $l=K$ where node 3 recovers the descriptions from the other nodes and generates its own ones. Node 1 recovers the last descriptions destined to it from nodes 2 and 3 but does not generate new ones. The same holds for node 2 who only recovers the descriptions generated by node 3 and thus terminating the information exchange procedure. At termination each node recovers an estimate of the other sources by using all decoded descriptions available from K rounds.

B. Sketch of the proof

1) Codebook generation: Consider the round $l\in[1:K]$ in node 1. Generate $2^{n\hat{R}^{(l)}_{1\to 23}}$ independent and identically distributed n-length codewords $U^n_{1\to 23,l}(m_{1\to 23,l}, m_{\mathcal{W}_{[1,l]}})$ according to the product probability measure:

$$\prod_{i=1}^n \Pr\left(u_{1\to 23,l,i}(m_{1\to 23,l}, m_{\mathcal{W}_{[1,l]}})|w_{[1,l],i}(m_{\mathcal{W}_{[1,l]}})\right)\ .$$

Index them with $m_{1\to 23,l}\in[1:2^{n\hat{R}^{(l)}_{1\to 23}}]$ and let $m_{\mathcal{W}_{[1,l]}}$ denote the indices of the descriptions $\mathcal{W}^n_{[1,l]}$ previously generated. Distribute them uniformly over $2^{nR^{(l)}_{1\to 23}}$ bins denoted by $\mathcal{B}_{1\to 23,l}\left(p_{1\to 23,l}, m_{\mathcal{W}_{[1,l]}}\right)$, and index them with $p_{1\to 23,l}\in[1:2^{nR^{(l)}_{1\to 23}}]$. Also generate $2^{n\hat{R}^{(l)}_{1\to 2}}$ and $2^{n\hat{R}_{1\to 3,l}}$ independent and identically distributed n-length codewords $U^n_{1\to 2,l}(m_{1\to 2,l}, m_{\mathcal{W}_{[2,l]}}, m_{\mathcal{V}_{[12,l,1]}})$, and index them with $m_{1\to 23,l}\in[1:2^{n\hat{R}^{(l)}_{1\to 2}}]$ and $U^n_{1\to 3,l}(m_{1\to 3,l}, m_{\mathcal{W}_{[2,l]}}, m_{\mathcal{V}_{[13,l,1]}})$ with $m_{1\to 3,l}\in[1:2^{n\hat{R}^{(l)}_{1\to 3}}]$ according to the product probability measures:

$$\begin{aligned}
&\prod_{i=1}^n \Pr\left(u_{1\to 2,l,i}(m_{1\to 2,l}, m_{\mathcal{W}_{[2,l]}}, m_{\mathcal{V}_{[12,l,1]}})\middle|w_{[2,l],i}(m_{\mathcal{W}_{[2,l]}}),\right.\\
&\qquad\qquad \left.v_{[12,l,1],i}(m_{\mathcal{V}_{[12,l,1]}})\right)\ ,\\
&\prod_{i=1}^n \Pr\left(u_{1\to 3,l,i}(m_{1\to 2,l}, m_{\mathcal{W}_{[2,l]}}, m_{\mathcal{V}_{[13,l,1]}})\middle|w_{[2,l],i}(m_{\mathcal{W}_{[2,l]}}),\right.\\
&\qquad\qquad \left.v_{[13,l,1],i}(m_{\mathcal{V}_{[13,l,1]}})\right)\ .
\end{aligned}$$

These codewords are distributed uniformly on $2^{nR^{(l)}_{1\to 2}}$ bins denoted by $\mathcal{B}_{1\to 2,l}\left(p_{1\to 2,l}, m_{\mathcal{W}_{[2,l]}}, m_{\mathcal{V}_{[12,l,1]}}\right)$, indexed with $p_{1\to 2,l}\in[1:2^{nR^{(l)}_{1\to 2}}]$ and on $2^{nR^{(l)}_{1\to 3}}$ bins denoted by $\mathcal{B}_{1\to 3,l}\left(p_{1\to 3,l}, m_{\mathcal{W}_{[2,l]}}, m_{\mathcal{V}_{[13,l,1]}}\right)$, indexed with $p_{1\to 3,l}\in[1:2^{nR^{(l)}_{1\to 3}}]$, respectively. The descriptions in nodes 2 and 3 are generated by following a similar procedure.

2) Encoding technique: Consider node 1 at round $l\in[1:K]$. Encoder 1 first looks for a codeword $u^n_{1\to 23,l}(m_{1\to 23,l}, \hat{m}_{\mathcal{W}_{[1,l]}})$ such that $\left(x^n_1, w^n_{[1,l]}(\hat{m}_{\mathcal{W}_{[1,l]}})u^n_{1\to 23,l}(m_{1\to 23,l}, \hat{m}_{\mathcal{W}_{[1,l]}})\right)$ is typical and where $\hat{m}_{\mathcal{W}_{[1,l]}}$ is the estimated index of $\mathcal{W}^n[1,l]$ at node 1 (notice that some components in this codeword are generated at node 1 and are perfectly known). If more than one codeword satisfied this condition, then we choose the one with the smallest index. Otherwise, if no such codeword exists, we choose an arbitrary index and declare an error. With the chosen index $m_{1\to 23,l}$, we determine the index $p_{1\to 23,l}$ of the bin $\mathcal{B}_{1\to 23,l}\left(p_{1\to 23,l}, \hat{m}_{\mathcal{W}_{[1,l]}}\right)$ to which $m_{1\to 23,l}$ belongs. Similarly, Encoder 1 looks for codewords $u^n_{1\to 2,l}(m_{1\to 2,l}, \hat{m}_{\mathcal{W}_{[2,l]}}, \hat{m}_{\mathcal{V}_{[12,l,1]}})$,

$u^n_{1\to 3,l}(m_{1\to 2,l}, \hat{m}_{\mathcal{W}_{[2,l]}}, \hat{m}_{\mathcal{V}_{[13,l,1]}})$ such that they are jointly typical with $\left(x^n_1, w^n_{[2,l]}(\hat{m}_{\mathcal{W}_{[2,l]}}), v^n_{[12,l,1]}(\hat{m}_{\mathcal{V}_{[12,l,1]}})\right)$ and $\left(x^n_1, w^n_{[2,l]}(\hat{m}_{\mathcal{W}_{[2,l]}}), v^n_{[13,l,1]}(\hat{m}_{\mathcal{V}_{[13,l,1]}})\right)$, respectively. The encoding procedure continues by finally determining $p_{1\to 2,l}$ and $p_{1\to 3,l}$ in a similar manner. Node 1 then transmits to node 2 and 3 the determined indices $(p_{1\to 23,l}, p_{1\to 2,l}, p_{1\to 3,l})$. The encoding in nodes 2 and 3 follows along the same lines.

3) Decoding technique: Consider round $l \in [1 : K+1]$ and node 2. The indices $(p_{1\to 23,l}, p_{3\to 12,l-1}, p_{1\to 2,l}, p_{3\to 2,l-1})$ are the ones relevant to node 2. Knowing this set of indices, node 2 aims to recover the exact values of $(m_{1\to 23,l}, m_{3\to 12,l-1}, m_{1\to 2,l}, m_{3\to 2,l-1})$. This is done through *successive decoding* where first, the common information indices are recovered by looking for the unique tuple of codewords $u^n_{1\to 23,l}(m_{1\to 23,l}, \hat{m}_{\mathcal{W}_{[1,l]}})$, $u^n_{3\to 12,l-1}(m_{3\to 23,l-1}, \hat{m}_{\mathcal{W}_{[3,l-1]}})$ which are jointly typical with $\left(x^n_2, w^n_{[3,l-1]}(\hat{m}_{\mathcal{W}_{[3,l-1]}})\right)$, and also belongs to the bins given by $p_{1\to 23,l}$ and $p_{3\to 12,l-1}$, and where $\hat{m}_{\mathcal{W}_{[3,l-1]}}$ is composed of indices of common information recovered and generated at node 2 at the previous rounds. If there are more than one pair of codewords, or none that satisfies this, we choose a predefined one and declare an error. After this, node 2 can recover the private information indices by looking at codewords $u^n_{1\to 2,l}(m_{1\to 2,l}, \hat{m}_{\mathcal{W}_{[2,l]}}, \hat{m}_{\mathcal{V}_{[12,l,1]}})$ and $u^n_{3\to 2,l-1}(m_{3\to 2,l-1}, \hat{m}_{\mathcal{W}_{[1,l]}}, \hat{m}_{\mathcal{V}_{[23,l-1,3]}})$ which are jointly typical with $(x^n_2, w^n_{[2,l]}(\hat{m}_{\mathcal{W}_{[2,l]}}), v^n_{[12,l,1]}(\hat{m}_{\mathcal{V}_{[12,l,1]}}),$ $v^n_{[23,l-1,3]}(\hat{m}_{\mathcal{V}_{[23,l-1,3]}}))$ and are in the bins given by $p_{1\to 2,l}$ and $p_{3\to 2,l-1}$. If there are more than one pair of codewords, or none that satisfies this, we choose a predefined one and declare an error.

4) Lossy reconstructions: After all rounds are accomplished, each node needs to estimate the source. For instance, node 1 reconstruct the source of node 2 by using:

$$\hat{x}_{12,i} = g_{12}\left(x_{1i}, v_{[12,K+1,1]i}, w_{[1,K+1]i}\right), \qquad (26)$$

for $i = 1, 2, \dots, n$ and similarly, for the source of node 3. Reconstruction at nodes 2 and 3 is done in the same way.

5) Error and distortion analysis: For lack of space we only provide a brief discussion. Denote with an upper case letter the true indices generated at the nodes, i.e. $M_{1\to 23,l}$. Consider the event $\mathcal{G}_l$ with $l \in [1 : K+1]$

$$\Big(X^n_1, X^n_2, X^n_3, \mathcal{W}^n_{[1,l]}(M_{\mathcal{W}_{[1,l]}}), \mathcal{V}^n_{[12,l,1]}(M_{\mathcal{V}_{[12,l,1]}}),$$
$$\mathcal{V}^n_{[13,l,1]}(M_{\mathcal{V}_{[13,l,1]}}), \mathcal{V}^n_{[23,l,2]}(M_{\mathcal{V}_{[23,l,3]}})\Big) \in \mathcal{T}^n_{\epsilon_l}, \epsilon_l > 0,$$

The key is to prove that $\mathbb{P}(\mathcal{G}_{K+1}) \to 1$ with $n \to \infty$. If this is the case it is straightforward to show that the average distortions satisfy the required fidelity constraints. Let $\mathcal{E}_l$ the event that during round l and in any node there is at least one error at the encoding and that at the decoding at least one of true indices are not recovered. It is not hard to see that

$$\mathbb{P}\left(\bar{\mathcal{G}}_{K+1}\right) \leq \sum_{l=1}^{K} \mathbb{P}\left(\mathcal{E}_l \cap \bar{\mathcal{G}}_l\right) + \mathbb{P}\left(\bar{\mathcal{G}}_1\right). \qquad (27)$$

When $n \to \infty$ it is clear that $\mathbb{P}\left(\bar{\mathcal{G}}_1\right) \to 0$. Similarly, using standard arguments, it can be shown that $\mathbb{P}\left(\mathcal{E}_l \cap \bar{\mathcal{G}}_l\right) \to 0, \forall l \in [1 : K]$. From the resulting rate equations we then need to eliminate the terms $\hat{R}^l_{i\to S}$ with $S \in \mathcal{C}(\mathcal{M})$, $i \notin S$ and $l \in [1 : K]$. This is accomplished through a Fourier-Motzkin elimination procedure [9]. In this manner the rates in equations (10)-(24) are obtained.

Remark 3: It is worth mentioning here that we constrained our scheme to use *successive decoding*, i.e., by recovering first common and then private descriptions. Although the best would be to allow *joint decoding*, only *successive decoding* makes possible our derivation of a closed-form rate-region.

Remark 4: The idea behind our derivation of the achievable region can be extended to any number M (> 3) of nodes in the network. This can be accomplished by generating a greater number of superimposed coding layers. First a layer of codes that generates descriptions intended to be decoded by all nodes. The next layer corresponding to all subsets of size $M-1$, etc, until we reach the final layer composed by codes that generate private descriptions for each of nodes. Again, *succesive decoding* is used at the nodes to recover the descriptions in these layers destined to them.

V. Summary and Discussion

We introduced the interactive three-terminal source coding problem and derived an inner bound to the rate-distortion region. Several previous results for interactive –as well non-interactive– lossy source coding problems were shown to be special case of this inner bound. Although not included (for lack of space) in this short ISIT submission format, an outer bound has also been derived which shows to be tight in several novel cases of interest [10].

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